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A Proof of the Ramanujan Hypothesis*

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A proof is given of Ramanujan's hypothesis on the coefficients of modular forms. The method of proof depends on Hecke's theory [1] of modular forms whose zeta functions admit Euler products. Basic to the theory is a commutative convolution algebra defined on the positive integers.

The elements of the algebra are the functions $f(n)$ of a positive integer variable n such that

$$\|f\| = \sum_{n=1}^{\infty} |f(n)|\, d(n) < \infty,$$

where $d(n)$ is the number of divisors of n. The convolution $h = f * g$ of two elements f and g of the algebra is the function $h(n)$ of positive integral n such that

$$h(n) = \sum_{rs=n} \sum_{k=1}^{\infty} f(kr)\, g(ks) = \sum_{r=1}^{\infty} \sum_{k|r,n} f(rn/k^2)\, g(k)$$

for every n. The function h so defined belongs to the algebra, and the inequality $\| h \| \leqslant \| f \| \, \| g \|$ is satisfied. Equality holds when f and g have nonnegative values since

$$\sum_{n=1}^{\infty} h(n)\, d(n) = \sum_{n=1}^{\infty} f(n) \sum_{r=1}^{\infty} g(r) \sum_{k|r,n} d(rn/k^2)$$

and since

$$d(r)\, d(n) = \sum_{k|r,n} d(rn/k^2).$$

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Convolution in the Hecke algebra is commutative and associative. Let $\delta_r(n)$ be the delta function defined by $\delta_r(n) = 1$ when $n = r$ and $\delta_r(n) = 0$ otherwise. The condition $g = \delta_r * f$ means that

$$g(n) = \sum_{k|r,n} f(rn/k^2)$$

for every positive integer n. The identity

$$\delta_r * \delta_n = \sum_{k|r,n} \delta_{rn/k^2}$$

holds for all positive integers r and n. The element δ_1 is a unit in the algebra. A conjugation is defined in the algebra by $g = f^*$ if $g(n) = \bar{f}(n)$ for every n.

A character χ of the Hecke algebra is a function $\chi(n)$ of positive integral n such that $\chi(1) = 1$, such that $|\chi(p)| \leqslant 2$ for every prime p, and such that

$$\chi(r)\,\chi(n) = \sum_{k|r,n} \chi(rn/k^2)$$

for all positive integers r and n. The function $\chi(n) = d(n)$ is the unique character with nonnegative values. Let $\omega(n)$ be any function of positive integral n such that $|\omega(n)| = 1$ for every n and such that $\omega(rn) = \omega(r)\,\omega(n)$ for all r and n. A corresponding character χ is defined by the equation

$$\chi(n) = \sum_{k|n} \omega(k)\,\omega(n/k)^{-1}.$$

Every character is of this form for some choice of ω. A character χ satisfies the inequality $|\chi(n)| \leqslant d(n)$ for every n. For any given character χ, there exists a corresponding linear functional $f \to \hat{f}(\chi)$ on the Hecke algebra defined by

$$\hat{f}(\chi) = \sum_{n=1}^{\infty} f(n)\,\chi(n).$$

This gives a homomorphism of the Hecke algebra onto the complex numbers. Every homomorphism of the algebra onto the complex numbers is of this form for a unique character χ. There is no nonzero element f of the algebra such that $\hat{f}(\chi) = 0$ for every character χ. A character χ is a real valued function.

Every element f of the Hecke algebra determines a convolution operator on the algebra defined by $g \to f * g$. The spectrum of the operator coincides with the set of values which the function $\hat{f}$ takes on the space of characters. An element f of the algebra is said to be nonnegative, $f \geqslant 0$, if $\hat{f}(\chi) \geqslant 0$ for every character χ. If f is an element of the algebra such that $f - a\delta_1 \geqslant 0$ for some positive number a, then $f = g * g^*$ for some element g of the algebra.

A representation space of the Hecke algebra is a Hilbert space $\mathscr{R}$ whose elements are functions $f(n)$ of a positive integral variable n, which contains $\delta_r * f$ whenever it contains f for every positive integer r, and which has these properties:

(1) The identity

$$\langle h * f, g\rangle_{\mathscr{R}} = \langle f, h^* * g\rangle_{\mathscr{R}}$$

holds for all elements f and g of $\mathscr{R}$ when h is a finite linear combination of delta functions.

(2) If f is any element of $\mathscr{R}$, then $h * f$ as an element of $\mathscr{R}$ depends continuously on h in the metric of the Hecke algebra when h is a finite linear combination of delta functions.

(3) The linear functional $f \to f(1)$ is continuous on $\mathscr{R}$.

Condition (2) allows the convolution $h * f$ to be defined by continuity for every element h of the Hecke algebra when f is in $\mathscr{R}$. Convolution by h is then a bounded transformation in $\mathscr{R}$ whose bound does not exceed the norm $\| h \|$ of h in the Hecke algebra. For if h is a given element of the algebra and if λ is a given number, $|\lambda| > \| h \|$, there exists a corresponding element k of the algebra such that $\lambda\bar{\lambda} = h * h^* + k * k^*$. It follows from (1) and (2) that the inequality

$$\langle h * f, h * f\rangle_{\mathscr{R}} \leqslant \lambda\bar{\lambda}\langle f, f\rangle_{\mathscr{R}}$$

holds for every element f of the algebra. By the arbitrariness of λ,

$$\| h * f \|_{\mathscr{R}} \leqslant \| h \| \, \| f \|_{\mathscr{R}} .$$

If $\mathscr{R}$ is a representation space, then by condition (3) there exists a unique element φ of $\mathscr{R}$ such that the identity $f(1) = \langle f, \varphi\rangle_{\mathscr{R}}$ holds for every element f of $\mathscr{R}$. Define a function $k(r, n)$ of two positive integer variables r and n by

$$k(r, n) = \sum_{s|r,n} \varphi(rn/s^2).$$

For each positive integer r, $k(r, n)$ is a function of n which coincides with $\delta_r * \varphi$ and so belongs to $\mathscr{R}$. If f is in $\mathscr{R}$ and if $g = \delta_r * f$, then condition (1) gives

$$\langle f, \delta_r * \varphi\rangle_{\mathscr{R}} = \langle g, \varphi\rangle_{\mathscr{R}} = g(1) = f(r).$$

It follows that the expression $k(r, n)$ is positive-definite in the sense that the inequality

$$\sum c_i \bar{c}_j k(n_i, n_j) \geqslant 0$$

holds whenever $n_1, \ldots, n_s$ are positive integers and $c_1, \ldots, c_s$ are corresponding complex numbers. Since convolution by δ_r is a bounded transformation in $\mathscr{R}$ whose bound is at most $\| \delta_r \| = d(r)$, the inequality

$$| \langle \delta_r * f, g \rangle_{\mathscr{R}} | \leqslant d(r) \| f \|_{\mathscr{R}} \| g \|_{\mathscr{R}}$$

holds for all elements f and g of $\mathscr{R}$. When $f = g = \varphi$, the inequality reads $| \varphi(r) | \leqslant \varphi(1)\, d(r)$. By Lemma 11 of [2], the given representation space $\mathscr{R}$ is uniquely determined by a knowledge of the corresponding function φ. The space is therefore denoted $\mathscr{R}(\varphi)$. The subscript $\mathscr{R}(\varphi)$ on inner products is abbreviated to φ.

Let $\varphi(n)$ be a given function of the positive integer variable n. Necessary conditions for the existence of a space $\mathscr{R}(\varphi)$ are positive-definiteness of the expression

$$k(r, n) = \sum_{s | r, n} \varphi(rn/s^2)$$

and boundedness of $\varphi(n)/d(n)$. These necessary conditions are also sufficient. For by Lemma 11 of [2], there exists a unique Hilbert space $\mathscr{R}$, whose elements are functions of a positive integer variable, such that $\delta_r * \varphi$ belongs to $\mathscr{R}$ for every positive integer r and such that the identity

$$f(r) = \langle f, \delta_r * \varphi \rangle_{\mathscr{R}}$$

holds for every element f of $\mathscr{R}$. It can be shown that $\mathscr{R}$ is a representation space of the Hecke algebra.

To do this, define a linear functional L on the algebra by

$$L(f) = \sum_{n=1}^{\infty} f(n)\, \varphi(n).$$

This is possible because $\varphi(n)/d(n)$ is a bounded function of n. Consider any elements f and g of the algebra which are finite linear combinations of delta functions. Then $f * \varphi$ and $g * \varphi$ belong to $\mathscr{R}$ and

$$\begin{aligned} L(f * g^*) &= \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} \sum_{k | r, n} f(n)\, \bar{g}(r)\, \varphi(rn/k^2) \\ &= \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} f(n)\, \bar{g}(r) \langle \delta_n * \varphi, \delta_r * \varphi \rangle_{\mathscr{R}} \\ &= \langle f * \varphi, g * \varphi \rangle_{\mathscr{R}} . \end{aligned}$$

It follows that $L(f) \geqslant 0$ whenever f is a nonnegative element of the algebra.

If f and g are finite linear combinations of delta functions, then $f * g$ is also a finite linear combination of delta functions. Since

$$\|f\|^2 g * g^* - f * g * f^* * g^*$$

is a nonnegative element of the algebra,

$$L(f * g * f^* * g^*) \leqslant \|f\|^2 L(g * g^*)$$

and

$$\|f * g * \varphi\|_{\mathscr{R}} \leqslant \|f\| \, \|g * \varphi\|_{\mathscr{R}} \, .$$

Note that the finite linear combinations of functions $\delta_n * \varphi$ are dense in $\mathscr{R}$. For $f(n) = 0$ whenever f is an element of $\mathscr{R}$ orthogonal to $\delta_n * \varphi$. The inequality

$$\|f * h\|_{\mathscr{R}} \leqslant \|f\| \, \|h\|_{\mathscr{R}}$$

follows by continuity for every element h of $\mathscr{R}$ when f is a finite linear combination of delta functions. It follows that $\mathscr{R} = \mathscr{R}(\varphi)$ is the required representation space.

Consider any function $\varphi(n)$ of positive integral n such that $\varphi(n)/d(n)$ is bounded. If f is an element of the Hecke algebra, then $\psi = f * \varphi$ is a function such that $\psi(n)/d(n)$ is bounded. The condition for the existence of a space $\mathscr{R}(\varphi)$ is positive-definiteness of the expression

$$k(r, n) = \sum_{s|r,n} \varphi(rn/s^2).$$

This is equivalent to the requirement that $\psi(1) \geqslant 0$ whenever f is a function of the form $f = g * g^*$ where g is a finite linear combination of delta functions. An equivalent condition is that $\psi(1) \geqslant 0$ whenever f is a nonnegative element of the Hecke algebra. Since a convolution of nonnegative elements is nonnegative, a space $\mathscr{R}(\psi)$ exists whenever $\psi = f * \varphi$ where $\mathscr{R}(\varphi)$ exists and f is a nonnegative element of the algebra.

A decomposition theory can be given for representation spaces. If a space $\mathscr{R}(\varphi)$ is contained in a space $\mathscr{R}(\psi)$ and if the inclusion does not increase norms, define

$$\|f\|_1^2 = \sup[\|f + g\|_\psi^2 - \|g\|_\varphi^2],$$

where the supremum is taken over all elements g of $\mathscr{R}(\varphi)$. The proof of Theorem 7 of [2] will show that the set $\mathscr{R}_1$ of all elements of finite 1-norm is a Hilbert space in the 1-norm. If f belongs to $\mathscr{R}(\varphi)$ and if g belongs to $\mathscr{R}_1$, then the inequality

$$\|h\|_\psi^2 \leqslant \|f\|_\varphi^2 + \|g\|_1^2$$

holds with $h = f + g$. Every element h of $\mathscr{R}(\psi)$ has a unique minimal decomposition $h = f + g$ with f in $\mathscr{R}(\varphi)$ and g in $\mathscr{R}_1$ such that equality holds. A straightforward use of the decomposition theory will show that a space $\mathscr{R}(\theta)$ exists, $\theta = \psi - \varphi$, and that it is equal isometrically to $\mathscr{R}_1$. A necessary and sufficient condition that the inclusion of $\mathscr{R}(\varphi)$ in $\mathscr{R}(\psi)$ be isometric is that the intersection of $\mathscr{R}(\varphi)$ and $\mathscr{R}(\theta)$ contain no nonzero element. If $\mathscr{R}(\varphi)$, $\mathscr{R}(\theta)$, and $\mathscr{R}(\psi)$ are given spaces such that $\varphi = \psi - \theta$, then $\mathscr{R}(\varphi)$ is contained in $\mathscr{R}(\psi)$ and the inclusion does not increase norms.

If $\mathscr{R}(\psi)$ is a space which is not one-dimensional, there exist spaces $\mathscr{R}(\varphi)$ and $\mathscr{R}(\theta)$ such that φ and θ are linearly independent and $\psi = \varphi + \theta$. For there exists at least one positive integer r such that ψ and $\delta_r * \psi$ are linearly independent. The required functions φ and θ are given by $\varphi = \frac{1}{2}\,\delta_r * \psi$ and $\theta = \psi - \frac{1}{2}\,\delta_r * \psi$.

If a space $\mathscr{R}(\varphi)$ is one-dimensional, φ and $\delta_r * \varphi$ are linearly dependent for every positive integer r. This is the case when, and only when, φ is a positive multiple of a character.

Consider the convex set of all functions φ such that a space $\mathscr{R}(\varphi)$ exists and such that $\varphi(1) = 1$. The set is compact in the topology of pointwise convergence. By the Kreĭn-Milman convexity theorem, it is the closed convex span of its extreme points. The extreme points of the set are the characters of the Hecke algebra.

For each positive integer n, let $\rho(n)$ be n times the product of the numbers $1 + 1/p$ taken over the distinct prime factors p of n. If $\mathscr{R}(\varphi)$ is a representation space of the Hecke algebra, the identity

$$\sqrt{n}\,\varphi(n) = \sum_{k^2 | n} \rho(n/k^2)\,\psi(n/k^2)$$

holds for a parameter function ψ which is bounded by one. By the Kreĭn-Milman theorem, it is sufficient to prove the existence of ψ in the case that $\varphi = \chi$ is a character. Let ω be a function such that $|\,\omega(n)| = 1$ for every n, such that $\omega(rn) = \omega(r)\,\omega(n)$ for all r and n, and such that

$$\chi(n) = \sum_{k | n} \omega(k)\,\omega(n/k)^{-1}.$$

The required function ψ is now obtained by a straightforward calculation. Parameter functions are used to define a multiplicative structure on representation spaces.

Let $\mathscr{R}(\varphi_1)$ and $\mathscr{R}(\varphi_2)$ be representation spaces of the Hecke algebra, and let ψ_1 and ψ_2 be the corresponding parameter functions. Then there exists a representation space $\mathscr{R}(\varphi)$ whose parameter function ψ is the product of ψ_1 and ψ_2. By the Kreĭn-Milman theorem, it is sufficient to show the existence of $\mathscr{R}(\varphi)$ in the case that $\varphi_1 = \chi_1$ and $\varphi_2 = \chi_2$ are characters of the Hecke algebra. By continuity it is sufficient to consider characters which have value two at all but a finite number of primes. In this case a straightforward calculation with the functions ω_1 and ω_2 will express φ as a convex combination of characters.

A simple example of a representation space is obtained when $\varphi = \delta_1$. The space is the set of all square summable functions f in the norm

$$\|f\|_\varphi^2 = \sum_{n=1}^{\infty} |f(n)|^2.$$

Other examples are obtained from the theory of modular forms.

The modular group $\Gamma(1)$ is the group of 2×2 matrices with integer entries and determinant one. The signature for the group is the unique homomorphism of the group into the complex numbers of absolute value one such that

$$\operatorname{sgn}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \exp(-\pi i/6).$$

Let ν and μ be integers. A modular form of order ν and signature (exponent) μ is a function $F(z)$, analytic in the upper halfplane, which satisfies the identity

$$F(z) = \frac{1}{(Cz + D)^{1+\nu}} \operatorname{sgn}^\mu \begin{pmatrix} A & B \\ C & D \end{pmatrix} F\left(\frac{Az + B}{Cz + D}\right)$$

for every element $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of the group. For each positive integer r, let $\Gamma(r)$ be the subgroup of $\Gamma(1)$ formed by those matrices whose lower left entry is divisible by r. An automorphic form of order ν and signature μ with respect to $\Gamma(r)$ is a function $F(z)$, analytic in the upper halfplane, which satisfies the same identity for every element of $\Gamma(r)$.

Two points w_1 and w_2 in the upper halfplane are said to be equivalent with respect to $\Gamma(r)$ if $w_2 = (Aw_1 + B)/(Cw_1 + D)$ for some element $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of $\Gamma(r)$. Note that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ belongs to $\Gamma(1)$ whenever $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ belongs to $\Gamma(1)$. This fact is useful in constructing a fundamental region for $\Gamma(1)$. A point w in the upper halfplane is said to be real with respect to $\Gamma(1)$ if the points w and $-\bar{w}$ are equivalent with respect to $\Gamma(1)$. If $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ belongs to $\Gamma(1)$, the equation $-\bar{w} = (Aw + B)/(Cw + D)$ has a nonreal solution when, and only when, the matrix has equal entries on the diagonal. When the lower left entry is zero, the matrix is of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \pm \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}$$

for an integer n. The solutions of the equation are the points on the vertical line through $\frac{1}{2}n$. When the lower left entry is not zero, the matrix is of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \pm \begin{pmatrix} -p & (p^2-1)/q \\ q & -p \end{pmatrix}$$

for relatively prime integers p and q, q positive and $p^2 - 1$ divisible by q. The solutions of the equation are the points on the circle of radius $1/q$ about the point p/q.

The points in the upper halfplane which are real with respect to $\Gamma(1)$ form a closed set whose complement is the union of its connected components. If $\binom{A\ B}{C\ D}$ is an element of $\Gamma(1)$, the transformation $z \to (Az + B)/(Cz + D)$ takes each component onto a component. Each component maps onto a different component when the matrix is not the identity matrix. Two components are said to be equivalent if one is obtained from the other in this way by an element of $\Gamma(1)$. The symmetry $z \to -\bar{z}$ takes each component onto a nonequivalent component. Two components are said to be conjugate if one is equivalent to the symmetric image of the other. Any two nonequivalent components are conjugate. A fundamental region for $\Gamma(1)$ is obtained on piecing together two conjugate components. An example is the set of points w in the upper halfplane such that $-\frac{1}{2} < \operatorname{Re} w < \frac{1}{2}$ and $|w| > 1$.

A fundamental region for $\Gamma(r)$ is obtained by piecing together fundamental regions for $\Gamma(1)$, one corresponding to each coset of $\Gamma(1)$ with respect to $\Gamma(r)$. Two elements $\binom{A_1\ B_1}{C_1\ D_1}$ and $\binom{A_2\ B_2}{C_2\ D_2}$ of $\Gamma(1)$ are said to be equivalent with respect to $\Gamma(r)$ if

$$\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$$

for some element $\binom{A\ B}{C\ D}$ of $\Gamma(r)$. The structure of the $\Gamma(r)$-equivalence classes of $\Gamma(1)$ is seen by considering a related equivalence relation on a different set. Let Γ be the set of matrices with integer entries and positive determinant. Two elements $\binom{A_1\ B_1}{C_1\ D_1}$ and $\binom{A_2\ B_2}{C_2\ D_2}$ of Γ are said to be equivalent with respect to $\Gamma(1)$ if the previous identity holds for some element $\binom{A\ B}{C\ D}$ of $\Gamma(1)$. Every element of Γ is equivalent with respect to $\Gamma(1)$ to a matrix which has zero as its lower left entry. Every element of Γ of determinant r whose entries have no nontrivial common factor can be written in the form

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for some elements $\binom{A\ B}{C\ D}$ and $\binom{A'\ B'}{C'\ D'}$ of $\Gamma(1)$. The number of equivalence classes of elements of Γ of determinant r whose entries have no nontrivial common factor is the sum, taken over the divisors k of r, of the number of integers modulo k which are relatively prime to k. This number is equal to the product $\rho(r)$ of r and the numbers $1 + 1/p$ where p is a prime factor of r. If $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ is an element of Γ of determinant r whose entries have no nontrivial common factor, then

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} 0 & -1 \\ r & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for some elements $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ of $\Gamma(1)$. The equivalence class of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with respect to $\Gamma(r)$ determines, and is uniquely determined by, the equivalence class of $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ with respect to $\Gamma(1)$. It follows that the index of $\Gamma(r)$ in Γ is equal to $\rho(r)$.

If $F(z)$ is an automorphic form of order ν and signature μ with respect to $\Gamma(r)$, the expression $y^{1+\nu} |F(z)|^2$ remains unchanged when z is replaced by $(Az + B)/(Cz + D)$ for a matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $\Gamma(r)$. The Petersson norm [3] of $F(z)$ is defined by integrating this expression with respect to the element $y^{-2}\, dx\, dy$ of hyperbolic area, which is invariant under all modular substitutions. The region of integration is a fundamental region $\Omega(r)$ for $\Gamma(r)$. The hyperbolic area of any such region is $\frac{1}{3} \pi\rho(r)$. Let $\mathscr{P}_\nu{}^\mu(r)$ be the set of automorphic forms $F(z)$ of order ν and signature μ with respect to $\Gamma(r)$ such that

$$\tfrac{1}{3} \pi\rho(r) \, \|F\|_\nu^2 = \iint_{\Omega(r)} |F(z)|^2 y^{\nu-1}\, dx\, dy < \infty.$$

The integral does not depend on the choice of fundamental region. The space $\mathscr{P}_\nu^\mu(r)$ is contained isometrically in the space $\mathscr{P}_\nu^\mu(s)$ when r divides s. If $F(z)$ is in $\mathscr{P}_\nu^\mu(r)$, its orthogonal projection $G(z)$ in $\mathscr{P}_\nu^\mu(1)$ is given by

$$\rho(r)\, G(z) = \sum \frac{1}{(Cz + D)^{1+\nu}} \operatorname{sgn}^\mu \begin{pmatrix} A & B \\ C & D \end{pmatrix} F\left(\frac{Az + B}{Cz + D}\right),$$

where summation is over all $\Gamma(r)$-equivalence classes of elements $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $\Gamma(1)$.

Note that the transformation $z \to -1/(rz)$ takes any fundamental region for $\Gamma(r)$ onto a fundamental region for $\Gamma(r)$. The function

$$G(z) = \frac{r^{\frac{1}{2}(1+\nu)}}{(rz)^{1+\nu}} F\left(\frac{-1}{rz}\right)$$

is an automorphic form of order ν and signature $r\mu$ with respect to $\Gamma(r)$ whenever $F(z)$ is an automorphic form of order ν and signature μ with respect to $\Gamma(r)$. The transformation $F(z) \to G(z)$ so defined is an isometry in the Petersson norm. Since the transformation is its own inverse when ν is odd and $r - 1$ is divisible by d, it is then self-adjoint. When ν is even and $r - 1$ is divisible by d, it is equal to minus its own adjoint. For each positive integer r

which is relatively prime to $12/d$, define a transformation $\Lambda(r)$ of $\mathcal{P}_\nu{}^\mu(1)$ into $\mathcal{P}_\nu^{r\mu}(1)$ by $\Lambda(r): F(z) \to G(z)$ if $G(z)$ is the orthogonal projection of

$$\frac{r^{\frac{1}{2}(1+\nu)}}{(rz)^{1+\nu}} F\left(\frac{-1}{rz}\right)$$

in $\mathcal{P}_\nu^{r\mu}(1)$. When ν is odd, the adjoint of the transformation $\Lambda(r) : \mathcal{P}_\nu{}^\mu(1) \to \mathcal{P}_\nu^{r\mu}(1)$ is the transformation $\Lambda(r)$: $\mathcal{P}_\nu^{r\mu}(1) \to \mathcal{P}_\nu{}^\mu(1)$. When ν is even, each transformation is minus the adjoint of the other. The rth Hecke operator $\Delta(r)$ is the transformation of $\mathcal{P}_\nu{}^\mu(1)$ into $\mathcal{P}_\nu^{r\mu}(1)$ defined by

$$\Delta(r) = \sum_{k^2 | r,n} \rho(r/k^2)\, \Lambda(r/k^2)\, \Lambda(1)^*,$$

where $\Lambda(1)^*$ is the adjoint of $\Lambda(1)$.

Some information about signatures is needed to compute Hecke operators. The cubes of the signatures of two elements of the modular group are equal if the matrices are congruent modulo four. The fourth powers of the signatures are equal if the matrices are congruent modulo three. The signature of any element of the modular group is a twelfth root of unity which is determined by a knowledge of its third and fourth powers. The signatures of the matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ are complex conjugates when $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is in the modular group. In extending signatures to elements of Γ, it is natural to define powers of the signature in some cases where the signature itself is not defined. Let d be a divisor of 12. If $\left(\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right)$ is an element of Γ whose determinant is congruent to one or minus one modulo $12/d$, define

$$\operatorname{sgn}^\mu \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \operatorname{sgn}^\mu \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

whenever μ is divisible by $12/d$, where $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is any element of Γ which is congruent modulo $12/d$ to $\left(\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The μ-th power of the signature of an element of Γ is defined implicitly whenever its determinant is relatively prime to $12/d$, where d is the greatest common divisor of μ and 12. The identity

$$\operatorname{sgn}^{\nu\mu} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \operatorname{sgn}^\mu \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \operatorname{sgn}^\mu \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$$

holds whenever it is meaningful for elements of Γ such that

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}.$$

if r is the determinant of the left factor.

A straightforward calculation will now show that the Hecke operators satisfy the identity

$$\Delta(r)\,\Delta(n) = \sum_{k|r,n} k\Delta(rn/k^2)$$

whenever it is meaningful. Assume that $\Delta(r) : F(z) \to G(z)$ where $F(z)$ is in $\mathscr{P}_\nu{}^\mu(1)$ and r is relatively prime to $12/d$. Then $G(z)$ is the element of $\mathscr{P}_\nu^{r\mu}(1)$ given by

$$G(z) = \sum \frac{r^{\frac{1}{2}(1+\nu)}}{(Cz+D)^{1+\nu}} \operatorname{sgn}^\mu \begin{pmatrix} A & B \\ C & D \end{pmatrix} F\left(\frac{Az+B}{Cz+D}\right)$$

where summation is over all $\Gamma(1)$-equivalence classes of elements $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of Γ of determinant r. Since $F(z)$ belongs to $\mathscr{P}_\nu{}^\mu(1)$, its Fourier series expansion in the upper halfplane is of the form

$$F(z) = \sum_{n=1}^{\infty} n^{\frac{1}{2}\nu} f(n) \exp(\pi i n d z/6),$$

where $f(n) = 0$ whenever $n - \mu/d$ is not divisible by $12/d$. If the analogous expansion is made for $G(z)$, a straightforward calculation will show that $g = \sqrt{r}\,\delta_r * f$.

Fundamental examples of modular forms are obtained from the functions

$$\lambda(z) = \exp(\pi i z/12) \prod_{n=1}^{\infty} [1 - \exp(2\pi i n z)],$$

$$\rho(z) = 1 + 240 \sum_{n=1}^{\infty} n^3 \exp(2\pi i n z)/[1 - \exp(2\pi i n z)],$$

and

$$\sigma(z) = 1 - 504 \sum_{n=1}^{\infty} n^5 \exp(2\pi i n z)/[1 - \exp(2\pi i n z)].$$

The functions $\lambda(z)^2$, $\rho(z)$, and $\sigma(z)$ are modular forms of order 0 and signature 1, order 3 and signature 0, and order 5 and signature 0. They satisfy the identity

$$\rho(z)^3 - \sigma(z)^2 = 1728\ \lambda(z)^{24}.$$

Every modular form of finite Petersson norm is a linear combination of products of powers of $\lambda(z)^2$, $\rho(z)$, and $\sigma(z)$. The dimension of $\mathscr{P}_\nu{}^\mu(1)$ is zero if $1 + \nu - \mu$ is odd. It is equal to one more than the greatest integer which is less than $\mu/12$ if $1 + \nu - \mu = 4\alpha + 6\beta$ where $\alpha = 0, 1,$ or 2 and $\beta = 0$ or 1.

Since the spaces $\mathscr{P}_\nu{}^\mu(1)$ are finite dimensional and since the Hecke operators $\Delta(r)$, $r-1$ divisible by d, are a commuting family of self-adjoint transformations in the space, the space admits an orthogonal basis whose elements are eigenfunctions of $\Delta(r)$ for every r. A conjecture of Ramanujan [4] states that the bound of $\Delta(r)$ is at most $\sqrt{r}\,d(r)$. The estimate is easily verified when $\nu = 0$ by computing the coefficients of $\lambda(z)^2$. An inductive procedure is now used to verify the estimate for every positive integer ν.

Tensor product spaces are used in making estimates of Hecke operators. For each positive integer r, let $\mathscr{P}_\nu^\mu(r) \otimes \mathscr{P}_\nu^\mu(r)^-$ be the set of finite sums of functions $F(z)\bar{G}(z)$ with $F(z)$ and $G(z)$ in $\mathscr{P}_\nu^\mu(r)$. An element $F(z)$ of the product space satisfies the identity

$$F(z) = \frac{1}{|Cz+D|^{2+2\nu}} F\left(\frac{Az+B}{Cz+D}\right)$$

for every element $\binom{A\ B}{C\ D}$ of $\Gamma(r)$. The Petersson norm of an element $F(z)$ of the space is defined by

$$\tfrac{1}{3}\pi\rho(r)\,\|F\|_{1+2\nu}^2 = \iint_{\Omega(r)} |F(z)|^2 y^{2\nu}\,dx\,dy$$

where integration is over a fundamental region $\Omega(r)$ for $\Gamma(r)$. The space $\mathscr{P}_\nu^\mu(r) \otimes \mathscr{P}_\nu^\mu(r)^-$ is contained isometrically in the space $\mathscr{P}_\nu^\mu(s) \otimes \mathscr{P}_\nu^\mu(s)^-$ when r divides s. The transformation

$$F(z) \to \frac{r^{1+\nu}}{|rz|^{2+2\nu}} F\left(\frac{-1}{rz}\right)$$

is an isometry of $\mathscr{P}_\nu{}^\mu(r) \otimes \mathscr{P}_\nu{}^\mu(r)^-$ into $\mathscr{P}_\nu^{r\mu}(r) \otimes \mathscr{P}_\nu^{r\mu}(r)^-$ when r is relatively prime to d. If $F(z)$ is in $\mathscr{P}_\nu{}^\mu(r) \otimes \mathscr{P}_\nu{}^\mu(r)^-$, an element $G(z)$ of the space is given by

$$\rho(r)\,G(z) = \sum \frac{1}{|Cz+D|^{2+2\nu}} F\left(\frac{Az+B}{Cz+D}\right),$$

where summation is over all $\Gamma(r)$-equivalence classes of elements $\binom{A\ B}{C\ D}$ of $\Gamma(1)$. The function $G(z)$ so obtained is the orthogonal projection of $F(z)$ into the set of elements $H(z)$ of $\mathscr{P}_\nu^\mu(r) \otimes \mathscr{P}_\nu^\mu(r)^-$ such that the identity

$$H(z) = \frac{1}{|Cz+D|^{2+2\nu}} H\left(\frac{Az+B}{Cz+D}\right)$$

holds for every element $\binom{A\ B}{C\ D}$ of $\Gamma(1)$. This set contains $\mathscr{P}_\nu^\mu(1) \otimes \mathscr{P}_\nu^\mu(1)^-$.

The tensor product space is suggested by an analogous construction for the space of square summable power series. Let $\mathscr{C}(z)$ be the Hilbert space of power series $f(z) = \sum a_n z^n$ with complex coefficients such that

$$\|f(z)\|^2 = \sum |a_n|^2 < \infty.$$

A related space is the space $\mathscr{C}(z) \otimes \mathscr{C}(z)^-$ of power series $f(z) = \sum a_{nr} z^n \bar{z}^r$ in z and $\bar{z}$ such that

$$\|f(z)\|^2 = \sum |a_{nr}|^2 < \infty.$$

Note that $f(z)\bar{g}(z)$ belongs to $\mathscr{C}(z) \otimes \mathscr{C}(z)^-$ whenever $f(z)$ and $g(z)$ belong to $\mathscr{C}(z)$. The finite sums of such products are dense in $\mathscr{C}(z) \otimes \mathscr{C}(z)^-$. The norm of the product space satisfies the identity

$$\langle f(z)\bar{g}(z), h(z)\bar{k}(z)\rangle = \langle f(z), h(z)\rangle \langle k(z), g(z)\rangle$$

for all elements $f(z)$, $g(z)$, $h(z)$, and $k(z)$ of $\mathscr{C}(z)$. It follows that a sum

$$f_1(z)\bar{g}_1(z) + \cdots + f_r(z)\bar{g}_r(z)$$

with $f_1(z),\dots, f_r(z)$ and $g_1(z),\dots, g_r(z)$ in $\mathscr{C}(z)$ can vanish identically only when the functions $f_1(z),\dots, f_r(z)$ are linearly dependent or the functions $g_1(z),\dots, g_r(z)$ are linearly dependent. To establish this, it is clearly sufficient to consider the case in which the functions $g_1(z),\dots, g_r(z)$ form an orthonormal set. In this case the stated result follows from the identity

$$\|f_1(z)\bar{g}_1(z) + \cdots + f_r(z)\bar{g}_r(z)\|^2 = \|f_1(z)\|^2 + \cdots + \|f_r(z)\|^2.$$

The same information can be given without reference to the norm of $\mathscr{C}(z)$. If $f_1(z),\dots, f_r(z)$ and $g_1(z),\dots, g_r(z)$ are functions analytic in a region such that

$$f_1(z)\bar{g}_1(z) + \cdots + f_r(z)\bar{g}_r(z) = 0,$$

then either the functions $f_1(z),\dots, f_r(z)$ are linearly dependent or the functions $g_1(z),\dots, g_r(z)$ are linearly dependent. This fact justifies the tensor product notation for the space $\mathscr{P}_\nu^\mu(r) \otimes \mathscr{P}_\nu^\mu(r)^-$.

The tensor product structure allows a new norm to be defined in the space $\mathscr{P}_\nu^\mu(r) \otimes \mathscr{P}_\nu^\mu(r)^-$. There exists a unique inner product $\langle P, Q\rangle_{\nu,\nu}$ defined on elements $P(z)$ and $Q(z)$ of $\mathscr{P}_\nu^\mu(r) \otimes \mathscr{P}_\nu^\mu(r)^-$ such that the identity

$$\langle F(z)\bar{G}(z), H(z)\bar{K}(z)\rangle_{\nu,\nu} = \langle F(z), H(z)\rangle_\nu \langle K(z), G(z)\rangle_\nu$$

holds for all elements $F(z), G(z)$, $H(z)$, and $K(z)$ of $\mathscr{P}_\nu^\mu(r)$. Some new spaces are needed to relate the tensor product norm to the Petersson norm in the tensor product space.

Let $\Gamma(r,r)$ be the set of elements of $\Gamma(1)$ which are congruent modulo r to the identity matrix. Then $\Gamma(r, r)$ is a normal subgroup of $\Gamma(1)$ which is contained in $\Gamma(r)$. The index of $\Gamma(r, r)$ in $\Gamma(r)$ is equal to r times the number of integers modulo r which are relatively prime to r. A fundamental region for $\Gamma(r, r)$ is obtained by piecing together fundamental regions for $\Gamma(r)$. The Petersson norm has an obvious generalization for automorphic forms of order ν with respect to $\Gamma(r, r)$. Let $\mathscr{P}_\nu{}^\mu(r, r)$ be the Hilbert space of automorphic forms of order ν and signature μ with respect to $\Gamma(r, r)$ which have finite Petersson norm. If $\binom{A\ B}{C\ D}$ is in the modular group, the transformation

$$z \to (Az + B)/(Cz + D)$$

takes any fundamental region for $\Gamma(r, r)$ onto a fundamental region for $\Gamma(r, r)$. The transformation

$$F(z) \to \frac{1}{(Cz + D)^{1+\nu}} F\left(\frac{Az + B}{Cz + D}\right)$$

takes automorphic forms of order ν and signature μ with respect to $\Gamma(r, r)$ into automorphic forms of order ν and signature $r\mu$ with respect to $\Gamma(r, r)$ and preserves the Petersson norm when r is relatively prime to d. The space $\mathscr{P}_\nu{}^\mu(r)$ is contained isometrically in the space $\mathscr{P}_\nu{}^\mu(r, r)$.

The space $\mathscr{P}_\nu^\mu(r, r) \otimes \mathscr{P}_\nu^\mu(r, r)^-$ is defined by the obvious analogy with the space $\mathscr{P}_\nu^\mu(r) \otimes \mathscr{P}_\nu^\mu(r)^-$, which it contains isometrically. The Petersson norm and the tensor product norm have obvious generalizations to the new tensor product space. For every element $\binom{A\ B}{C\ D}$ of the modular group, the transformation

$$F(z) \to \frac{1}{|\, Cz + D\,|^{2+2\nu}} F\left(\frac{Az + B}{Cz + D}\right)$$

takes $\mathscr{P}_\nu{}^\mu(r, r) \otimes \mathscr{P}_\nu{}^\mu(r, r)^-$ into $\mathscr{P}_\nu^{r\mu}(r, r) \otimes \mathscr{P}_\nu^{r\mu}(r, r)^-$ when r is relatively prime to d. It is isometric in the Petersson norm and in the tensor product norm.

This result is used to obtain information about the tensor product norm in $\mathscr{P}_\nu^\mu(r) \otimes \mathscr{P}_\nu^\mu(r)^-$. If $F(z)$ is in the space, consider the element $G(z)$ of the space given by

$$\rho(r)\, G(z) = \sum \frac{1}{|\, Cz + D\,|^{2+2\nu}} F\left(\frac{Az + B}{Cz + D}\right),$$

where summation is over all $\Gamma(r)$-equivalence classes of elements $\binom{A\ B}{C\ D}$ of $\Gamma(1)$. Then $G(z)$ is the orthogonal projection, in the tensor product norm, of $F(z)$ into the set of elements $H(z)$ of $\mathscr{P}_\nu^\mu(r) \otimes \mathscr{P}_\nu^\mu(r)^-$ such that the identity

$$H(z) = \frac{1}{|\, Cz + D\,|^{2+2\nu}} H\left(\frac{Az + B}{Cz + D}\right)$$

holds for every element $\binom{A\ B}{C\ D}$ of $\Gamma(1)$. Note that the transformation

$$F(z) \to \frac{r^{1+\nu}}{|\, rz\,|^{2+2\nu}} F\left(\frac{-1}{rz}\right)$$

is an isometry of the space $\mathscr{P}_\nu{}^\mu(r) \otimes \mathscr{P}_\nu{}^\mu(r)^-$ into the space $\mathscr{P}_\nu^{r\mu}(r) \otimes \mathscr{P}_\nu^{r\mu}(r)^-$ in the tensor product norm when r is relatively prime to d.

Let $\mathscr{M}_\nu{}^\mu(r)$ be the Hilbert space of functions generated by $\mathscr{P}_\nu{}^\mu(1) \otimes \mathscr{P}_\nu{}^\mu(1)^-$ under the action of the transformation

$$\Delta(r) : F(z) \to \sum \frac{r^{1+\nu}}{|\, Cz + D\,|^{2+2\nu}} F\left(\frac{Az + B}{Cz + D}\right),$$

$r - 1$ divisible by d. Summation is over all $\Gamma(1)$-equivalence classes of elements $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of Γ of determinant r. The identity

$$F(z) = \frac{1}{|\, Cz + D\,|^{2+2\nu}} F\left(\frac{Az + B}{Cz + D}\right)$$

holds for every element $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of $\Gamma(1)$ if $F(z)$ is in $\mathscr{M}_\nu{}^\mu(1)$. The adjoint of the transformation $\Delta(r) : \mathscr{M}_\nu{}^\mu(1) \to \mathscr{M}_\nu^{r\mu}(1)$ is the transformation

$$\Delta(r) : \mathscr{M}_\nu^{r\mu}(1) \to \mathscr{M}_\nu{}^\mu(1)$$

if the spaces are considered in the Petersson norm or the tensor product norm. It follows that the bound of these transformations does not depend on the choice of norm. The tensor product norm is used to estimate the bound when the Ramanujan hypothesis holds in $\mathscr{P}_\nu^\mu(1)$.

Let ν be held fixed, and let $F(z)$ and $G(z)$ be given elements of $\mathscr{P}_\nu^\mu(1)$. Let $\varphi(n)$ and $\psi(n)$ be the functions of positive integral n, $n - 1$ divisible by d, defined by

$$\sqrt{n}\, \varphi(n) = \langle \Delta(n) F, F \rangle_\nu$$

and

$$\sqrt{n}\, \psi(n) = \langle \Delta(n)\, G, G \rangle_\nu\,.$$

Since the identity

$$\Delta(r)\, \Delta(n) = \sum_{k | r,n} k\Delta(rn/k^2)$$

holds whenever r and n are relatively prime to d, the expression $\sum_{k|r,n} \varphi(rn/k^2)$ is positive-definite on the set of positive integers which are congruent to one modulo d.

Let $\mathscr{R}_d(\varphi)$ be the unique Hilbert space, whose elements are functions of positive integral n, $n - 1$ divisible by d, such that $\delta_r * \varphi$ belongs to the space whenever $r - 1$ is divisible by d, and such that the identity

$$f(r) = \langle f, \delta_r * \varphi \rangle_\varphi$$

holds for every element f of the space. There exists a unique isometry $u \to U(z)$ of $\mathcal{R}_d(\varphi)$ into $\mathcal{P}_\nu{}^\mu(1)$ such that

$$\sqrt{r}\, u(r) = \langle \Delta(r)\, U, F\rangle_\nu$$

whenever $r - 1$ is divisible by d. Since the Ramanujan hypothesis is assumed in $\mathcal{P}_\nu{}^\mu(1)$ and since the action of $\Delta(r)/\sqrt{r}$ in $\mathcal{P}_\nu{}^\mu(1)$ corresponds to convolution by δ_r in $\mathcal{R}_d(\varphi)$, the bound of convolution by δ_r in $\mathcal{R}_d(\varphi)$ is at most $d(r)$ when $r - 1$ is divisible by d. The space $\mathcal{R}_d(\psi)$ is defined in the same way with $F(z)$ replaced by $G(z)$. Let $\theta(n)$ be the function of positive integral n, $n - 1$ divisible by d, defined by

$$\sqrt{n}\, \theta(n) = \langle \Delta(n)\, H, H\rangle_{\nu,\nu}$$

where $H(z) = F(z)\, \bar{G}(z)$. Let $\alpha(n)$ and $\beta(n)$ be the functions of positive integral n, $n - 1$ divisible by d, such that

$$\sqrt{n}\, \varphi(n) = \sum_{k^2 | n} \rho(n/k^2)\, \alpha(n/k^2)$$

and

$$\sqrt{n}\, \psi(n) = \sum_{k^2 | n} \rho(n/k^2)\, \beta(n/k^2)$$

whenever $n - 1$ is divisible by d. Then the identities

$$\left\langle \frac{n^{1/2(1+\nu)}}{(Cz + D)^{1+\nu}}\, F\left(\frac{Az + B}{Cz + D}\right), F(z)\right\rangle_\nu = \alpha(n)\, \| F \|_\nu^2$$

and

$$\left\langle \frac{n^{1/2(1+\nu)}}{(Cz + D)^{1+\nu}}\, G\left(\frac{Az + B}{Cz + D}\right), G(z)\right\rangle_\nu = \beta(n)\, \| G \|_\nu^2$$

hold for every element $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of Γ of determinant n. It follows that the identities

$$\left\langle \frac{n^{1+\nu}}{| Cz + D |^{2+2\nu}}\, H\left(\frac{Az + B}{Cz + D}\right), H(z)\right\rangle_{\nu,\nu} = \alpha(n)\, \beta(n)\, \| H \|_{\nu,\nu}^2$$

and

$$\sqrt{n}\, \theta(n) = \sum_{k^2 | n} \rho(n/k^2)\, \alpha(n/k^2)\, \beta(n/k^2)$$

hold whenever $n - 1$ is divisible by d.

Let $\mathcal{R}_d(\theta)$ be the unique Hilbert space whose elements are functions of positive integral n, $n - 1$ divisible by d, such that $\delta_r * \theta$ belongs to the space whenever $r - 1$ is divisible by d and such that the identity $f(r) = \langle f, \delta_r * \theta \rangle_\theta$ holds for every element f of the space. Since $\varphi(n)/d(n)$ and $\psi(n)/d(n)$ are bounded functions of n, $\theta(n)/d(n)$ is a bounded function of n. It follows that convolution by δ_r is an everywhere defined transformation with bound $d(r)$ in $\mathscr{R}_d(\theta)$ when $r - 1$ is divisible by d. There exists a unique isometry $u \to U(z)$ of $\mathscr{R}_d(\theta)$ into $\mathscr{M}_\nu{}^\mu(1)$ such that

$$\sqrt{r}\, u(r) = \langle \Delta(r)\, U, H \rangle_{\nu,\nu}$$

whenever $r - 1$ is divisible by d. Since convolution by δ_r in $\mathscr{R}_d(\theta)$ corresponds to the action of $\Delta(r)/\sqrt{r}$ in $\mathscr{M}_\nu{}^\mu(1)$, the bound of $\Delta(r)$ is at most $\sqrt{r}\, d(r)$ in the closed subspace of $\mathscr{M}_\nu{}^\mu(1)$ generated by $H(z)$. By the arbitrariness of $F(z)$ and $G(z)$, the bound of $\Delta(r)$ is at most $\sqrt{r}\, d(r)$ in $\mathscr{M}_\nu{}^\mu(1)$ when $r - 1$ is divisible by d. It follows that the bound of the transformation $\Delta(r) : \mathscr{M}_\nu{}^\mu(1) \to \mathscr{M}_\nu^{r\mu}(1)$ is at most $\sqrt{r}\, d(r)$ when r is relatively prime to d.

The following argument can now be used to verify the Ramanujan hypothesis for $\Delta(r)$ in $\mathscr{P}_{1+\nu}^{1+\mu}(1)$ when $r - 1$ is relatively prime to twelve and the Ramanujan hypothesis is known in $\mathscr{P}_\nu{}^\mu(1)$. When μ is not divisible by twelve, multiplication by $\lambda(z)$ takes the space $\mathscr{P}_\nu{}^\mu(1)$ onto the space $\mathscr{P}_{1+\nu}^{1+\mu}(1)$. When μ is divisible by twelve, it takes $\mathscr{P}_\nu{}^\mu(1)$ onto the set of elements of $\mathscr{P}_{1+\nu}^{1+\mu}(1)$ which have leading Fourier coefficient equal to zero. The space $\mathscr{P}_{1+\nu}^{1+\mu}(1)$ is clearly generated by this set under the action of the Hecke operators. Since these spaces are finite dimensional, there exists a finite constant $\kappa(\nu, \mu)$ such that the inequality

$$\| F\bar{F} \|_{1+2\nu} \leqslant \kappa(\nu, \mu)^2 \| \lambda F \|_{1+\nu}^2 / \| \lambda\bar{\lambda} \|_1$$

holds for every element $F(z)$ of $\mathscr{P}_\nu{}^\mu(1)$. If r is relatively prime to twelve and if $\Delta(r) : \lambda(z) F(z) \to G(z)$, then

$$G(z) = \sum \frac{1}{(Cz + D)^{2+\nu}} \operatorname{sgn}^{1+\mu} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \lambda \left(\frac{Az + B}{Cz + D} \right) F \left(\frac{Az + B}{Cz + D} \right)$$

where the sum is taken over all $\Gamma(1)$-equivalence classes of elements $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of Γ of determinant r. It follows that

$$2y^{1/2(2+\nu)} \mid G(z) \mid$$
$$\leqslant \sum \frac{y}{\mid Cz + D \mid^2} \left| \lambda \left(\frac{Az + B}{Cz + D} \right) \right|^2 + \sum \frac{y^{1+\nu}}{\mid Cz + D \mid^{2+2\nu}} \left| F \left(\frac{Az + B}{Cz + D} \right) \right|^2$$

where summation is over the same set of equivalence classes. By the triangle inequality,

$$2 \| G \|_{1+\nu} \leqslant \| \Delta(r) \lambda\bar{\lambda} \|_1 + \| \Delta(r) F\bar{F} \|_{1+2\nu} \, .$$

Since the bound of $\Delta(r)$ is at most $\sqrt{r}\, d(r)$ in $\mathscr{P}_0{}^1(1)$ and in $\mathscr{P}_\nu{}^\mu(1)$, it follows that

$$2\| G \|_{1+\nu} \leqslant \sqrt{r}\, d(r) \| \lambda\bar{\lambda} \|_1 + \sqrt{r}\, d(r)\, \kappa(\nu, \mu)^2 \| \lambda F \|_{1+\nu}^2 / \| \lambda\bar{\lambda} \|_1 .$$

Since the same conclusion holds with $F(z)$ replaced by any constant multiple of itself,

$$\| G \|_{1+\nu} \leqslant \kappa(\nu, \mu)\, \sqrt{r}\, d(r) \| \lambda F \|_{1+\nu} .$$

By the arbitrariness of $F(z)$, the bound of $\Delta(r)$ in $\mathscr{P}_{1+\nu}^{1+\mu}(r)$ is at most $\kappa(\nu, \mu)\, \sqrt{r}\, d(r)$. By the arbitrariness of r, the bound of $\Delta(r)$ in $\mathscr{P}_{1+\nu}^{1+\mu}(r)$ is at most $\sqrt{r}\, d(r)$.

Since the Ramanujan hypothesis holds in $\mathscr{P}_0{}^1(1)$, an inductive argument will now show that the bound of $\Delta(r)$ in $\mathscr{P}_\nu{}^\mu(1)$ is at most $\sqrt{r}\, d(r)$ whenever $\mu = 1 + \nu$ if r is relatively prime to twelve. If $\nu = 4\alpha + 6\beta$ where $\alpha = 0$, 1, or 2 and $\beta = 0$ or 1, then the space $\mathscr{P}_\nu{}^1(1)$ is one-dimensional and the space $\mathscr{P}_\nu^{1+\nu}(1)$ contains a nonzero element. It follows that there exists a positive integer n, $n - 1 - \nu$ divisible by the greatest common divisor of $1 + \nu$ and twelve, such that $\Delta(n)$ takes $\mathscr{P}_\nu^{1+\nu}(1)$ onto $\mathscr{P}_\nu{}^1(1)$. Since the bound of $\Delta(r)$ is at most $\sqrt{r}\, d(r)$ in $\mathscr{P}_\nu^{1+\nu}(1)$, it is at most $\sqrt{r}\, d(r)$ in $\mathscr{P}_\nu{}^1(1)$. An inductive argument will now show that the bound of $\Delta(r)$ is at most $\sqrt{r}\, d(r)$ in every space $\mathscr{P}_\nu{}^\mu(1)$ when r is relatively prime to twelve. When $F(z)$ belongs to $\mathscr{P}_\nu{}^\mu(1)$ and r is not relatively prime to $12/d$, the functions

$$F(z) \qquad \text{and} \qquad \frac{r^{1/2(1+\nu)}}{(rz)^{1+\nu}} F\left(\frac{-1}{rz}\right)$$

are automorphic forms with respect to $\Gamma(r)$ for different signatures, and so are orthogonal. A similar argument will show that the bound of $\Delta(r)$ in $\mathscr{P}_\nu{}^\mu(1)$ is at most $\sqrt{r}\, d(r)$ whenever r is relatively prime to $12/d$.

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